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AN APPROACH–EVASION GAME OF TWO CONTROLLABLE OBJECTS WITH RESTRICTED MANOEUVRABILITY†

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Two controllable objects of restricted manoeuvrability, moving in a plane so that one tries to approach the other, are treated in a game-theoretic context. An optimal positional strategy is constructed for the pursuer, guaranteeing, for certain constraints on the problem parameters, approach to within a minimum distance from the evader at a certain time, which is not specified in advance. A computer is used to construct the domain of the evader's initial positions for which the pursuer's optimal strategy proposed in the paper guarantees coincidence capture of the evader within a time not exceeding a prescribed value. Other conditions are imposed on the game parameters so that the pursuer's strategy as constructed here will guarantee the minimum distance between the objects for the entire duration of the game. The paper is related to [1–8].

1. THE EQUATIONS OF MOTION AND THE PAYOFF FUNCTIONAL

SUPPOSE that objects P (the pursuer) and E (the evader) are moving at constant velocities in the XOY plane (Fig. 1). The equations of motion of the objects and the constraints on their controls are the same as in the well-known "game of two cars" problem [1]

$$x_i^{\cdot} = V_i \sin\theta_i,\quad y_i^{\cdot} = V_i\cos\theta_i,\quad \theta_i^{\cdot} = (V_i / R_i)\varphi_i,\quad |\varphi_i| \leqslant 1 \tag{1.1}$$

where V_i is a constant velocity, R_i is the minimum radius of curvature of the trajectory, θ_i is the angle between the Y axis and the vector V_i, φ_i is a scalar control, C_i is the centre of curvature of the trajectory, and $i=1$ corresponds to the object P and $i=2$ to the object E.

We shall assume throughout that system (1.1) satisfies the following condition ($\partial\theta_i$ is a fairly small quantity)

$$\theta_i = \theta_i^0 + \delta\theta_i \tag{1.2}$$

In view of (1.2), we conclude that the motion of P and E is governed by a system of linear equations

$$\begin{gathered} x_i^{\cdot} = V_i(\sin\theta_i^0 + z_i\cos\theta_i^0),\quad y_i^{\cdot} = V_i(\cos\theta_i^0 - z_i\sin\theta_i^0) \\ z_i^{\cdot} = (V_i / R_i)\varphi_i,\quad |\varphi_i| \leqslant 1\quad (z_i = \delta\theta_i) \end{gathered} \tag{1.3}$$

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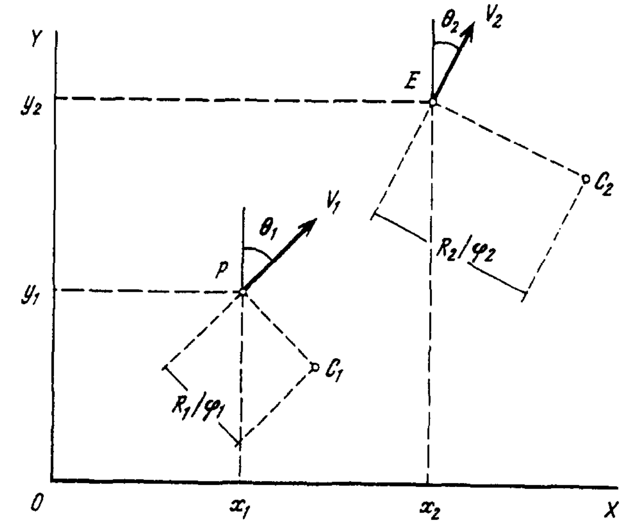

FIG. 1.

The payoff functional is defined by the relation

$$\gamma = \min_{t_0 < t < T} \{[x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2\}^{1/2} \tag{1.4}$$

where T is the time at which the game ends. The game is considered in a time interval $[t_0, T]$, where $T > t_0$ is fairly long. The pursuer P minimizes the payoff (1.4) of the game, and the evader maximizes it. The initial positions of P and E are given by the vectors $\{x_i^0, y_i^0, z_i^0 = 0\}$ $(i = 1, 2)$.

2. STATEMENT OF THE PROBLEM

For the differential game (1.3), (1.4), we wish to find relations among the parameters V_i, R_i, θ_i^0 $(i = 1, 2)$ and to construct a positional strategy $u(t, x_1, y_1, z_1, x_2, y_2, z_2)$: $[t_0, T] \times R^3 \times R^3 \to \{\varphi_1\}$ for the pursuer P, which will guarantee his approach, subject to the satisfaction of the above relations, to within a minimum distance from the evader E at some time $t \in [t_0, T]$, where

$$\{\varphi_1\} = \{\varphi_1 \in R_1 \colon |\varphi_1| \leq 1\}$$

3. REACHABLE DOMAINS OF P AND E

In the XOY plane in which the objects P and E are moving, let us consider the reachable domains of the objects corresponding to a time $t = T$ $(T > t_0)$ and initial positions $\{t_0, x_i^0, y_i^0, z_i^0 = 0\}$ $(i = 1, 2)$ of P and E, respectively.

It has been shown [4] that the reachable domains $G^{(1)}(t, T)$ and $G^{(2)}(t, T)$ of P and E in problem (1.2)–(1.5) are straight-line segments A_1B_1 and A_2B_2 with centres at points O_1 and O_2, orthogonal to the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, where

$$\mathbf{n}_i = \{V_i(T - t)\sin\theta_i^0;\ V_i(T - t)\cos\theta_i^0\}$$

Put

$$r_i(t, T) = |O_iA_i| = |O_iB_i|$$

Integrating Eqs (1.3) for $\varphi_1 = +1$ or $\varphi_1 = -1$ ($\varphi_2 = +1$ or $\varphi_2 = -1$), we obtain

$$r_i(t, T) = (V_i(T - t))^2/(2R_i)$$

4. AUXILIARY CONSTRUCTIONS. TRANSFORMATION OF COORDINATES

To simplify what follows, we will transform the system of coordinates, placing the new origin at the initial position of P and pointing the OY axis along the latter's velocity vector; the OX axis will be to the right of V_1. Thus, the position of E in the new system of coordinates will be defined by a vector (x_2^*, y_2^*, θ_2^*), where

$$\theta_2^* = \theta_2^0 - \theta_1^0, \quad x_2^* = \rho\cos\theta_*, \quad y_2^* = \rho\sin\theta_*, \quad \rho = \sqrt{(x_2^0 - x_1^0)^2 + (y_2^0 - y_1^0)^2} \tag{4.1}$$
$$\theta_* = \varphi + \theta_1, \quad \varphi = \operatorname{arctg}[(y_2^0 - y_1^0)/(x_2^0 - x_1^0)]$$

Henceforth, for simplicity, we shall omit the asterisk in the notation for the position of E, also renaming the controls φ_i ($i = 1, 2$) of the objects as u for P and υ for E, respectively. The equations of motion (1.3) in the new system of coordinates become:
for P

$$\dot{x}_1 = V_1 z_1, \quad \dot{y}_1 = V_1, \quad \dot{z}_1 = (V_1/R_1)u, \quad |u| \leqslant 1 \tag{4.2}$$

for E

$$\dot{x}_2 = V_2(\sin\theta_2 + z_2\cos\theta_2), \quad \dot{y}_2 = V_2(\cos\theta_2 - z_2\sin\theta_2)$$
$$\dot{z}_2 = (V_2/R_2)\upsilon, \quad |\upsilon| \leqslant 1 \tag{4.3}$$

The expression for the payoff functional (1.4) remains unchanged. A typical configuration of the objects P and E and their reachable domains at some time $t = T$, where $T > t_0$, is shown in Fig. 2.

5. INTEGRATION OF THE SYSTEMS OF EQUATIONS OF MOTION FOR P AND E

Assuming that the controls of P and E are constant over $[t_0, T]$, we can integrate systems (4.2) and (4.3). Then the dynamics of P will be described by the equations

$$x_1 = uV_1^2(T - t_0)^2/(2R_1), \quad y_1 = V_1(T - t_0), \quad z_1 = uV_1(T - t_0)/R_1 \tag{5.1}$$

and that of E by

$$x_2 = x_2^0 + V_2(T - t_0)\sin\theta_2 + [\upsilon V_2^2(T - t_0)^2/(2R_2)]\cos\theta_2$$
$$y_2 = y_2^0 + V_2(T - t_0)\cos\theta_2 - [\upsilon V_2^2(T - t_0)^2/(2R_2)]\sin\theta_2 \tag{5.2}$$
$$z_2 = \upsilon V_2(T - t_0)/R_2$$

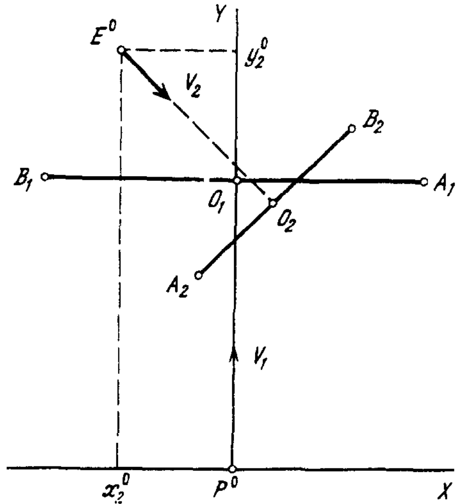

FIG. 2.

We fix arbitrary positions $\{t_1, x_i^1, y_i^1, z_i^1\}$ $(i=1, 2)$ of P and E, respectively, where $x_i^1 = x_i(t_1)$, $y_i^1 = y_i(t_1)$, $z_i^1 = z_i(t_1)$. Let us consider the reachable domains $G^{(1)}(t_1, x_1^1, y_1^1, z_1^1, T)$ and $G^{(2)}(t_1, x_2^1, y_2^1, z_2^1, T)$ for increasing values of $T > t_1$. The coordinates of the objects at time $t = T$, on condition that their positions at time $t = t_1$ were $\{t_1, x_i^1, y_i^1, z_i^1\}$ $(i = 1, 2)$, respectively, may be computed as follows:

for P

$$x_1 = x_1^1 + V_1 z_1^1 (T - t_1) + \upsilon V_1^2 (T - t_1)^2 / 2R_1, \quad y_1 = y_1^1 + V_1 (T - t_1)$$
$$z_1 = z_1^1 + u V_1 (T - t_1) / R_1 \tag{5.3}$$

for E

$$x_2 = x_2^1 + V_2 (T - t_1)(\sin\theta_2 + z_2^1 \cos\theta_2) + [\upsilon V_2^2 (T - t_1)^2 / (2R_2)] \cos\theta_2$$
$$y_2 = y_2^1 + V_2 (T - t_1)(\cos\theta_2 - z_2^1 \sin\theta_2) - [\upsilon V_2^2 (T - t_1)^2 / (2R_2)] \sin\theta_2 \tag{5.4}$$
$$z_2 = z_2^1 + \upsilon V_2 (T - t_1) / R_2$$

6. THE CONDITIONS UNDER WHICH THE REACHABLE DOMAIN OF E WILL BE COVERED BY THAT OF P

We will first define what is meant by "covering". Suppose that at some time $t = t_1$ $(t_0 \leq t_1)$ the objects P and E are located at positions $\{t_1, x_1^1, y_1^1, z_1^1\}$ and $\{t_1, x_2^1, y_2^1, z_2^1\}$, respectively. Consider the reachable domains of P and E. Let us assume that there are times $T_1 > t_1$ and $T_2 > T_1$ at which the reachable domains of P and E have points in common: at time $t = T_1$ $(T_1 > t_1)$ the coordinates of one of the extreme points of the reachable domain of E coincide with those of some extreme point of the reachable domain of P; and at time $t = T_2$ $(T_2 > T_1)$ the reachable domain of P will contain another extreme point of the reachable domain of E. When the reachable domains of P and E stand in such a relation at times $t = T_1$ and $t = T_2$, we shall say that the reachable domain of E is covered by that of P. Conditions relating the parameters of the objects P and E whose reachable domains are in a covering situation will be called covering conditions (CC).

7. COVERING CONDITIONS FOR DIFFERENT VALUES OF θ_2

Depending on the value of the angle θ_2 $(0 \leqslant \theta_2 \leqslant 2\pi)$, we shall consider four possible cases of the game (4.2), (4.3), (1.4) and find CC for each.

We shall assume that the following two inequalities hold

$$V_1 > V_2(\cos\theta_2^0 - z_2^1 \sin\theta_2^0) \tag{7.1}$$

$$[V_1 + V_2(z_2^1 \sin\theta_2^0 - \cos\theta_2^0)]^2 \geqslant 2[V_2^2(y_2^1 - y_1^1)/(2R_2)]\sin\theta_2^0 \tag{7.2}$$

7.1. *First case.* $0 < \theta_2^0 \leqslant \pi/2$ (*Fig.* 3)

The times at which extreme points of the reachable domain of E reach the straight lines $y = y_1(T)$ are the roots of the quadratic equation

$$\upsilon a(T - t_1)^2 + b(T - t_1) - c = 0 \tag{7.3}$$

$$(a = [V_2^2/(2R_2)]\sin\theta_2^0, \;\; b = V_1 + V_2(z_2^1 \sin\theta_2^0 - \cos\theta_2^0), \;\; c = y_2^1 - y_1^1 \text{ and } a \geqslant 0, \; c > 0)$$

They are

$$T_1 = t_1 + (-b + \Delta_+)/(2a) \;\; (\upsilon = +1), \qquad T_2 = t_1 + (b - \Delta_-)/(2a) < T_1 \;\; (\upsilon = -1)$$

$$(\Delta_\pm = \sqrt{b^2 \pm 4ac}) \tag{7.4}$$

It can be shown that conditions (7.1) and (7.2) guarantee the existence of times $T = T_i$ $(i = 1, 2)$.

The following conditions are sufficient for a covering situation to exist

$$x_1(T_1) = x_2(T_1), \qquad x_1(T_2) \leqslant x_2(T_2)$$

These conditions may be written as

$$a_1(T_1 - t_1)^2 + b_1(T_1 - t_1) + c_1 = 0 \tag{7.5}$$

$$-a_1(T_2 - t_1)^2 + b_1(T_2 - t_1) + c_1 \geqslant 0 \tag{7.6}$$

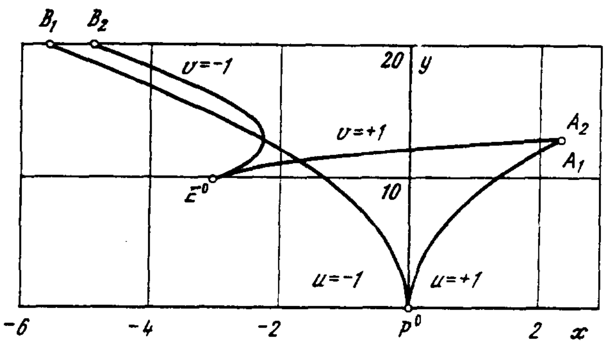

FIG. 3.

where

$$a_1 = [V_2^2/(2R_2)]\cos\theta_2^0 - V_1^2/(2R_1)$$
$$b_1 = V_2(\sin\theta_2^0 + z_2^1\cos\theta_2^0) - V_1 z_1^1, \quad c_1 = x_2^1 - x_1^1$$

Substitute the expression for c_1 obtained from Eq. (7.5) into inequality (7.6), together with the already determined values of T_1 and T_2. Remembering that $b>0$, we obtain $-a_1b+b_1a \geqslant 0$. Substituting the values of a, b, a_1 and b_1, putting

$$z_1^1 = (V_1/R_1)(t_1-t_0)u, \quad z_2^1 = (V_2/R_2)(t_1-t_0)\upsilon$$

and noting that P and E use controls $u=+1$ and $\upsilon=+1$ to reach the points $A_1(T_1)$, $A_2(T_1)$, respectively, and controls $u=-1$ and $\upsilon=-1$ to reach $B_1(T_2)$, $B_2(T_2)$, respectively, we can write inequality (7.6), which defines the relationship between the parameters of the players P and E, in the form

$$[V_1^2/(2R_1)](V_1 - V_2\cos\theta_2^0) - [V_2^2/(2R_2)](V_1\cos\theta_2^0 - V_2) \geqslant 0 \tag{7.7}$$

Obviously, this inequality is independent of the positions of P and E at time $t=t_1$. Hence it follows that, henceforth, to establish conditions like (7.7), we need only examine the initial positions of P and E, that is, we may assume that $t_1=t_0$. Then our notation will be

$$a = [V_2^2/(2R_2)]\sin\theta_2^0, \quad b = V_1 - V_2\cos\theta_2^0, \quad c = y_2^0 \quad (a \geqslant 0, \quad c>0)$$
$$a_1 = [V_2^2/(2R_2)]\cos\theta_2^0 - V_1^2/(2R_1), \quad b_1 = V_2\sin\theta_2^0, \quad c_1 = x_2^0$$

The sufficient conditions (7.1) and (7.2) for the existence of times T_1 and T_2, when $t=t_0$, may be written in the new notation as

$$\begin{aligned} &V_1 \geqslant V_2\cos\theta_2 \\ &V_1 \geqslant V_2(\cos\theta_2 + \sqrt{2y_2^0\sin\theta_2/R_2}) \end{aligned} \tag{7.8}$$

Thus, taken together, inequalities (7.7), (7.8) and Eq. (7.5) constitute CC.

Note that, whatever the value of θ_2, the computation of the values of b, c and b_1, c_1 will be the same. In addition, Eq. (7.5) and inequalities (7.8) are common to all other cases of the different values of θ_2. Our task is now, therefore, to find replacements for inequality (7.7) in these cases.

7.2. *Second case.* $\pi/2 \leqslant \theta_2^0 < \pi$ (*Fig.* 4)

Reasoning as before, we use a quadratic equation of the form (7.3), where $a>0$, $b>0$ and $c>0$, to find the times at which the extreme points of the reachable domain of E hit the straight lines $y=y_1(T_i)$ $(i=1, 2)$. Here $T=T_1$ is obtained for E moving with control $\upsilon=+1$, and $t=T_2$ for E with control $\upsilon=-1$. The parameter a is determined as in the first case, but

$$a_1 = [V_2^2/(2R_2)]\cos\theta_2^0 + [V_1^2/(2R_1)]$$

The inequality for the x-coordinate of the objects in the CC will be in this case

$$x_1(T_2) \geqslant x_2(T_2) \tag{7.9}$$

Proceeding as in the previous case, we obtain (7.9) in the form

$$-a_1b + b_1a \leqslant 0$$

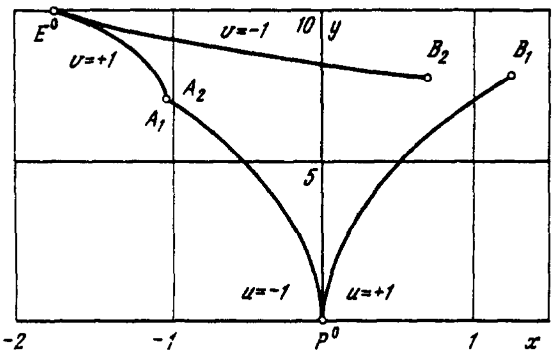

FIG. 4.

Since P and E use controls $u=-1$ and $\upsilon=+1$ to reach the points $A_1(T_1)$, $A_2(T_1)$, respectively, and controls $u=+1$ and $\upsilon=-1$ to reach $B_1(T_2)$, $B_2(T_2)$, respectively, we obtain the following form of inequality (7.9) for the CC

$$[V_1^2/(2R_1)](V_2\cos\theta_2^0-V_1)+[V_1\ /(2R_2)](V_2-V_1\cos\theta_2^0)\leqslant 0 \tag{7.10}$$

7.3. *Third case.* $\pi<\theta_2^0\leqslant 3\pi/2$ (*Fig.* 5)

All the reasoning is analogous to that in the previous cases, except that

$$a=-[V_2^2/(2R_2)]\sin\theta_2^0,\quad a_1=-[V_2^2/(2R_2)]\cos\theta_2^0-V_1^2/(2R_1)$$

The times $T=T_1$ and $T=T_2$ at which the extreme points of the reachable domain of E hit the straight lines $y=y_i(T_i)$ $(i=1, 2)$ are found from the quadratic equation obtained from (7.3) by replacing υ by $-\upsilon$.

The inequality for the x-coordinates of P and E in the CC is analogous to (7.6). Since P and E use controls $u=+1$ and $\upsilon=-1$ to reach the points $A_1(T_1)$, $A_2(T_1)$, respectively, and controls $u=-1$ and $\upsilon=+1$ to reach $B_1(T_2)$, $B_2(T_2)$, respectively, we obtain a condition relating the parameters of the objects, which is identical with (7.10).

7.4. *Fourth case.* $3\pi/2<\theta_2^0<2\pi$ (*Fig.* 6)

The times $t=T_1$ and $t=T_2$ at which the extreme points of the reachable domain of E hit the straight lines $y=y_i(T_i)$ $(i=1, 2)$ are found from the quadratic equation obtained from (7.3) by replacing υ by $-\upsilon$.

For the reachable domain of P to cover that of E at times $t=T_1$ and $t=T_2$, it will suffice that, instead of conditions (7.5), (7.8), an analogue of condition (7.6) should hold with

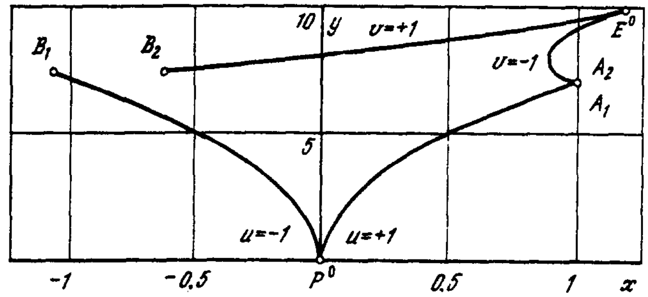

FIG. 5.

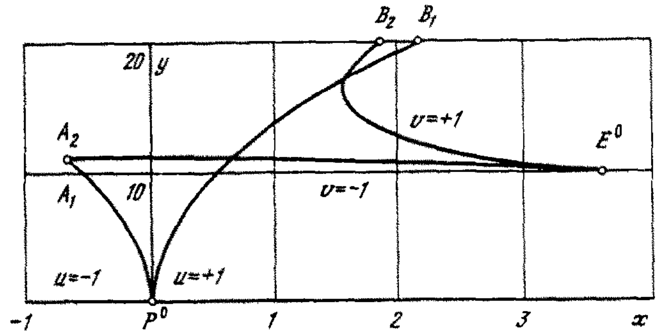

FIG. 6.

$$a_1 = -[V_2^2/(2R_2)]\cos\theta_2^0 + V_1^2/(2R_1)$$

Since P and E use controls $u=-1$ and $\upsilon=-1$ to reach the points $A_1(T_1)$, $A_2(T_1)$, respectively, and controls $u=+1$ and $\upsilon=+1$ to reach $B_1(T_2)$, $B_2(T_2)$, respectively, we obtain a condition for the x-coordinates of the objects in the CC, relating the players' parameters, which is identical with (7.7).

Remark 1. When $\theta_2=0$ and $\theta_2=\pi$, we have $T_1=T_2$. In that case problem (4.2), (4.3), (1.4) is a game with fixed time $T=T_1=T_2$ and terminal functional. This problem was considered in [4].

8. POSITIONAL STRATEGY FOR P

The conditions imposed on the parameters of the game (4.2), (4.3), (1.4) mean that the following pairs of inequalities cannot hold simultaneously

$$x_1(T_1) < x_2(T_1) \text{ and } x_1(T_2) > x_2(T_2) \quad \text{for} \quad 0 \leqslant \theta_2 < \pi/2 \text{ and } \pi \leqslant \theta_2 \leqslant 3\pi/2 \tag{8.1}$$

$$x_1(T_1) > x_2(T_1) \text{ and } x_1(T_2) < x_2(T_2) \quad \text{for} \quad \pi/2 \leqslant \theta_2 < \pi \text{ and } 3\pi/2 < \theta_2 < 2\pi \tag{8.2}$$

Positions of the game (4.2), (4.3), (1.4) for which inequality (7.7) or (7.10) is satisfied (depending on the value of θ_2) will be called regular positions. Positions of the game for which simultaneously

$$x_1(T_1) = x_2(T_1) \quad \text{and} \quad x_1(T_2) = x_2(T_2) \tag{8.3}$$

belong to the singular set S. Such positions may arise in this game, say, if the parameters of the objects are such that inequalities (7.7) and (7.10) become equalities.

Thus, for the above positions of the game (4.2), (4.3) and (1.4), the strategy of P is

$$u(t, x_1, y_1, z_1, x_2, y_2, z_2) = \begin{cases} \operatorname{sign}([x_2(T_2) - x_1(T_2)] + [x_2(T_1) - x_1(T_1)]), \\ \text{if inequality (8.1) or (8.2) holds;} \\ [-1,\ +1] \text{ if neither of inequalities (8.1)} \\ \text{and (8.2) holds or if (8.3) is true} \end{cases} \tag{8.4}$$

Let us assume that the object E, considered over the time interval $t_0 \leqslant t \leqslant T$, uses one of its extreme controls ($\upsilon=-1$ or $\upsilon=+1$). It is obvious that in these cases the strategy u of P ensures coincidence capture of E at a time $t=T_1$ or $t=T_2$, that is, the objects' coordinates coincide at those times. It can be shown that if E, considered over the time interval $t_0 \leqslant t \leqslant T_2$, chooses an arbitrary control with $|\upsilon| \leqslant 1$, capture will occur at a time T such that $T_1 \leqslant T \leqslant T_2$. For these cases, the payoff functional (1.4) becomes a coincidence functional

$$\gamma = \{[x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2\}^{1/2} = 0$$

If the position of the game with $t = t_0$ belongs to S, it can be shown that the strategy u of P will enable him to approach E to within as small a distance $\varepsilon > 0$ as desired.

9. HYPOTHETICAL MISMATCH FUNCTION

Let us consider a case in which inequality (7.7) or (7.10) of the CC (depending on the value of θ_2) fails to hold. Fix arbitrary positions $\{t, x_i, y_i, z_i\}$ of P and E, where $t > t_0$, and consider their reachable domains $G^{(1)}(t, T)$ and $G^{(2)}(t, T)$ at times $T = T_i$ $(i = 1, 2)$ when the extreme points of the domains lie on a single straight line $y = y_1(T_i)$ $(i = 1, 1)$. Put

$$\varepsilon_i(t) = |\, x_2(T_i) - x_1(T_i)\, | \tag{9.1}$$

We define the hypothetical mismatch function of the problem by the expression

$$\varepsilon(t) = \max[\varepsilon_1(t), \varepsilon_2(t)] \tag{9.2}$$

Positions of the game (4.2), (4.3), (1.4) for which the maximum in (9.2) is attained for one of the functions $\varepsilon_i(t)$ belong to the regular domain of the game. Positions for which $\varepsilon_1(t) = \varepsilon_2(t) = 0$ belong to the singular set S.

10. THE VALUE OF THE GAME

It can be shown that, for any positions of the game (4.2), (4.3), (1.4), whatever the value of the angle θ_2, if either of inequalities (7.7) or (7.10) (in the CC) holds, then $\varepsilon(t)$ will be u-stable [2] (this follows directly from the definition of u-stability).

It follows from the u-stability of $\varepsilon(t)$ that it is the value function of the game (4.2), (4.3), (1.4). This assertion, and the fact that the strategy (8.4) guarantees player P the result

$$\varepsilon_0 = \varepsilon(t_0) \geqslant \varepsilon(t) \quad \text{for} \quad t > t_0 \tag{10.1}$$

in the regular case and the same result (apart from arbitrary small $\varepsilon > 0$) for initial positions belonging to the singular set S, imply that (8.4) is an optimal pursuer's strategy.

11. CONSTRUCTION OF THE SET OF POSITIONS Q OF E FOR WHICH THE ENCOUNTER WITH P IS UNAVOIDABLE (FIXED PARAMETER VALUES OF BOTH OBJECTS)

Suppose we are given parameter values V_1, R_1 and V_2, R_2, $\theta_2 = \theta_2^0$ $(0 \leqslant \theta_2^0 \leqslant 2\pi)$ of P and E, respectively, as well as the initial position of P (at the origin). Assume, moreover, that the game is in a covering situation , so that the CC are satisfied for the given parameter values. Fix some arbitrary time $t = T_1$.

We wish to determine a position $E^0(x_2^0, y_2^0)$ of E at time $t_0 = 0$, in the set of positions Q, such that E cannot avoid P's approach.

Put

$$y_2^* = R(V_1 - V_2 \cos\theta_2^0)^2 / (2V_2 \sin\theta_2^0)$$

By (7.8), we may assume that the initial coordinate $y_2 = y_2^0$ of E satisfies the inequality

$$y_2^0 \leq y_2^* \tag{11.1}$$

As shown above, the reachable domains of P and E at time $t = T_1$ are straight-line segments A_iB_i orthogonal to the vectors $\mathbf{n}_i = \mathbf{V}_iT_1$ $(i = 1, 2)$, respectively. The coordinates of the points A_i and B_i $(i = 1, 2)$ can be computed using formulae (5.1) and (5.2).

At $t = T_1$, since we have a covering situation, the coordinates of the extreme points of the reachable domains of P and E are identical

$$x_1(T_1) = x_2(T_1), \qquad y_1(T_1) = y_2(T_1) \tag{11.2}$$

Using the CC, let us find $E(x_2^0, y_2^0)$ at time $t_0 = 0$. To that end, we first construct the reachable domain of E corresponding to $t = T_1$, which is a segment of a straight line through the common point of the reachable domains of P and E. We then drop a perpendicular on to the midpoint O_2 of the segment and, in the sense opposite to $\mathbf{n}_2$, mark off a segment O_2E equal to V_2T_1. The coordinates of the point E thus obtained are the required (x_2^0, y_2^0).

Now, varying the parameter T_1 from $T_1 = 0$ to $T_1 = T^*$, where T^* is the limiting value corresponding to $y_2 = y_2^*$, we construct a curve representing the set Q of initial positions of E from which it cannot avoid encounter with P when the parameter values of both objects take the above values. The curve Q is shown dashed in Fig. 7. One sees that the Q thus constructed is described by Eqs (5.2), where $x_2 = x_2(T_1)$, $y_2 = y_2(T_1)$ (we are using (11.2)).

12. CONSTRUCTION OF THE DOMAIN OF INITIAL POSITIONS FOR E FROM WHICH IT CANNOT AVOID CAPTURE BY P (FIXED PARAMETER VALUES OF E)

For given parameter values of E (velocity V_2, radius of curvature R_2 and inclination θ_2 of the velocity vector $\mathbf{V}_2$ to OY) and initial coordinate y_2^0 satisfying the constraint (11.1), we can use the CC to find the corresponding parameters of P. Inequalities (7.8) and (7.13) yield the velocity V_1. Depending on the value of θ_2, we now use (7.7) or (7.10) to determine the maximum possible radius of curvature R_1 of the pursuer's trajectory. Considering the maximum and minimum R_1, we then find the corresponding times T_1 at which (7.5) holds, from which we can compute the corresponding initial x-coordinates of E.

Running through all possible values of the initial y-coordinates of E that satisfy condition (11.1), we use the algorithm outlined above to determine, for each such value, the corresponding x-coordinate x_2^0. As a result, given parameters of $E(V_2, R_2, \theta_2)$, the velocity V_1 of P

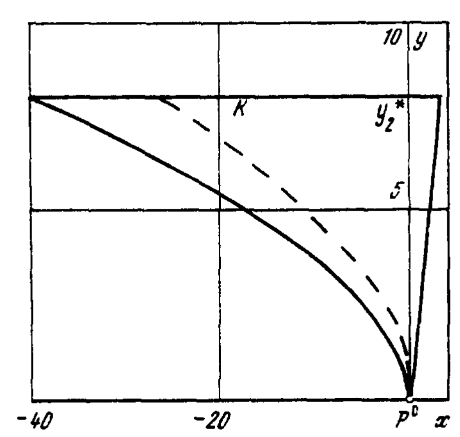

FIG. 7.

and two extreme values of its radius of curvature ($R^{\min}$, $R^{\max}$), we obtain two curves, representing the boundaries of the domain of initial positions of E. For initial positions of E between these curves, fixed parameter values of E and the parameter values of P derived from the CC, the approach–evasion game will end with the capture of E (either coincidence or ε-contact). One of these domains, for angles in the interval $270° < \theta_2 < 360°$ and parameters of P and E satisfying the CC, is shown in Fig. 7. This domain of initial positions, denoted by K, is a curvilinear triangle in the plane. Note that this domain K is the union of all the curves Q constructed in Sec. 11.

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